REPRESENTABILITY THEOREMS, UP TO HOMOTOPY

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Abstract. We prove two representability theorems, up to homotopy, for presheaves taking values in a closed symmetric combinatorial model category $\mathcal{V}$. The first theorem resembles the Freyd representability theorem, the second theorem is closer to the Brown representability theorem. As an application we discuss a recognition principle for mapping spaces.

1. Introduction

It is a classical question in homotopy theory whether for a given space $X \in \mathcal{T}\mathrm{op}_*$ there exists a space $Y \in \mathcal{T}\mathrm{op}_*$ such that $X \simeq \mathrm{hom}(S^1, Y)$. Several solutions to this question have emerged, beginning with Sugawara's work in [26]. The approach proposed by Stasheff in [25], Boardman and Vogt in [5], and May in [20] are known nowadays as operadic, while Segal's loop space machine (see [24]), is closer to Lawvere's notion of an algebraic theory. Later on, the canonical delooping machine by Badzioch, Chung, and Voronov (see [2]) provided a simplicial algebraic theory $\mathfrak{T}_n$, $n \geq 0$ allowing for the recognition of an n-fold loop space as a homotopy algebra over $\mathfrak{T}_n$.

It is natural to ask whether there exist similar recognition principles for mapping spaces of the form $X = \mathrm{hom}(A, Y)$ for $A \in \mathcal{T}\mathrm{op}_*$ other than S^n? When $A = S^2 \vee S^3$, for example, there is no simplicial algebraic theory $\mathfrak{T}$ such that all spaces of the form $X = \mathrm{hom}(S^2 \vee S^3, Y) = \Omega^2 Y \times \Omega^3 Y$ are homotopy algebras over $\mathfrak{T}$ (see [2, p. 2]). More generally, if a space A has rational homology in more than one positive dimension, there is no simplicial algebraic theory $\mathfrak{T}$ such that all the spaces of the form $X = \mathrm{hom}(A, Y)$ have the structure of homotopy algebras over $\mathfrak{T}$ (see [3]).

In this paper we suggest using a larger category than an algebraic theory for the recognition of arbitrary mapping spaces, up to homotopy. This category will be closed under arbitrary homotopy colimits, unlike an algebraic theory which is closed only under finite (co)products. The minimal subcategory of spaces containing A and closed under the homotopy colimits is denoted $C(A)$. This approach was first introduced by Badzioch, Dorabiała, and the first author in [1], where an attempt to limit the homotopy colimits involved was made. We take a different approach here, and consider the functors defined on the large subcategory of spaces $C(A)$. Given a space $X \in \mathcal{T}\mathrm{op}_*$, suppose there exists a functor $F\colon C(A) \to \mathcal{T}\mathrm{op}_*$ taking homotopy colimits to homotopy limits and satisfying $F(A) \simeq X$. The question of whether there exists a space Y such that $X \simeq \mathrm{hom}(A, Y)$ for some Y is equivalent to the question of representability of the functor F, up to homotopy.

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Theorems about representability of functors are naturally divided into two main types: Freyd and Brown representability theorems. In both cases some exactness condition for the functor under consideration is necessary. Theorems of Freyd type use a set-theoretical assumption about the functor: for example, the solution set condition, or accessibility (see [14, 3, Ex. G,J], [17, 4.84], and [22, 1.3]). Theorems of Brown type use set-theoretical assumptions about the domain category, such as the existence of sufficiently many compact objects (see [6], [21], [18], and [13]).

In this paper we address the question of representability (up to homotopy) of functors taking values in a closed symmetric combinatorial model category. After some technical preliminaries in Section 2, we prove the Freyd version of representability up to homotopy in Section 3. The solution set condition is replaced by the requirement that the functor be small. Theorem 3.1 generalizes [17, Theorem 4.84] to functors defined on a $\mathcal{V}$-model category, rather than just a $\mathcal{V}$-category. At the same time, it generalizes a result the first author on the representability of small contravariant functors from spaces to spaces (see [8]).

The Brown version of representability up to homotopy is proved in Section 4. The set-theoretical condition concerning the domain category is local presentability. In other words, we show that for any $\mathcal{V}$-presheaf H defined on a combinatorial $\mathcal{V}$-model category $\mathcal{M}$ and taking homotopy colimits to homotopy limits, there is a fibrant object $Y \in \mathcal{M}$ and a natural transformation $h\colon H(-) \to \hom(-, Y)$, which is a weak equivalence for every cofibrant $X \in \mathcal{M}$. A similar theorem for functors taking values in simplicial sets was proved by Jardine in [16]. However, the conditions required for the Brown representability, up to homotopy, to hold are formulated for the homotopy category of the model category and do not allow for an easy verification in an arbitrary combinatorial model category.

In Section 5 we provide an example of a non-small presheaf, defined on a non-combinatorial model category, which is not representable up to homotopy, This shows that representability theorems are not tautological.

In Section 6 we interpret Brown representability up to homotopy as a recognition principle for mapping spaces for an arbitrary space A (rather than just S^n).

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2. (Model) Categorical preliminaries

For every closed symmetric monoidal combinatorial model category $\mathcal{V}$ and $\mathcal{V}$-model category $\mathcal{M}$ (not necessarily combinatorial), we can consider the category of small presheaves $\mathcal{V}^{\mathcal{M}^{\mathrm{op}}}$. This is a $\mathcal{V}$-category of functors, which are left Kan extensions from small subcategory of $\mathcal{M}$. The category $\mathcal{V}^{\mathcal{M}^{\mathrm{op}}}$ is cocomplete by [17, 5.34]. Since $\mathcal{V}$ is a combinatorial model category, it is in particular locally presentable. Therefore, the category of small presheaves $\mathcal{V}^{\mathcal{M}^{\mathrm{op}}}$ is also complete by [10]. For a $\mathcal{V}$-category $\mathcal{C}$ we denote by $\mathcal{C}_0$ the underlying category of $\mathcal{C}$ (enriched only in Set).

Definition 2.1. A natural transformation $f\colon F \to G$ in $\left(\mathcal{V}^{\mathcal{M}^{\mathrm{op}}}\right)_0$ is called a *cofibrant-projective weak equivalence* (respectively, a *cofibrant-projective fibration*) if for all cofibrant $M \in \mathcal{M}$, the induced map $f_M\colon F(M) \to G(M)$ is a weak equivalence (respectively, a fibration). The notion of a *projective weak equivalence* (respectively, a *projective fibration*) in $\left(\mathcal{V}^{\mathcal{M}^{\mathrm{op}}}\right)_0$ is a particular case of the cofibrant-projective analog, when all objects of $\mathcal{M}$ are cofibrant – e.g., for the trivial model structure on $\mathcal{M}$. If (cofibrant-)projective fibrations and weak equivalences give rise to a model structure on the category of small presheaves, this model structure is called *(cofibrant-)projective.*

First of all, we would like to establish the existence of the cofibrant-projective model structure on $\left(\mathcal{V}^{\mathcal{M}^{\mathrm{op}}}\right)_0$. For a simplicial model category $\mathcal{M}$, with $\mathcal{V} = \mathcal{S}$ (the category of simplicial sets), this was proven in [9, 2.8]. For a combinatorial model category $\mathcal{M}^{op}$, this was proven in [4, 3.6]. But the case of contravariant small functors from a combinatorial model category to $\mathcal{V}$ is not covered by the previous results.

Condition 2.2. *Every trivial fibration in $\mathcal{V}$ is an effective epimorphism in $\mathcal{V}_0$ (cf. [23, II, p. 4.1]).*

Basic examples of categories satisfying this condition are (pointed) simplicial sets, spectra, and chain complexes.

Theorem 2.3. *Let $\mathcal{V}$ be a closed symmetric monoidal model category satisfying condition 2.2, and $\mathcal{M}$ a $\mathcal{V}$-model category. The category of small functors $\left(\mathcal{V}^{\mathcal{M}^{\mathrm{op}}}\right)_0$ may be equipped with the cofibrant-projective model structure. Moreover, $\mathcal{V}^{\mathcal{M}^{\mathrm{op}}}$ becomes a $\mathcal{V}$-model category.*

Proof. Let I and J be the classes of generating cofibrations and generating trivial cofibrations in $\mathcal{V}$, and let $\mathcal{M}_{\mathrm{cof}}$ denote the subcategory of cofibrant objects of $\mathcal{M}$.

The cofibrant-projective model structure on the category of small presheaves $\mathcal{V}^{\mathcal{M}^{\mathrm{op}}}$ is generated by the following classes of maps.

$$\mathcal{I} = \{\, R_M \otimes i |\, I \ni i\colon A \hookrightarrow B,\, M \in \mathcal{M}_{\mathrm{cof}}\}$$

and

$$\mathcal{J} = \{\, R_M \otimes j |\, J \ni j\colon U \overset{\sim}{\hookrightarrow} V,\, M \in \mathcal{M}_{\mathrm{cof}}\}$$

where R_M is the representable functor $X \mapsto \hom(X, M) \in \mathcal{V}$.

It suffices to verify that these two classes of maps admit the generalized small object argument (see [7]). More specifically, we need to show that $\mathcal{I}$ and $\mathcal{J}$ are locally small. In other words, for any map $f\colon X \to Y$ we need to find a *set* $\mathcal{W}$ of maps in $\mathcal{I}$-cof (respectively, $\mathcal{J}$-cof), such that every morphism of maps $R_M \otimes i \to f$ factors through an element in $\mathcal{W}$. By adjunction, it is sufficient to find a set of cofibrant objects $\mathcal{U}$, such that every map $R_M \to X^A \times_{Y^A} Y^B$ factors though an element of $\mathcal{U}$.

Consider the functor $F = X^A \times_{Y^A} Y^B$. Like any small functor, $F\colon \mathcal{M}^{\mathrm{op}} \to \mathcal{V}$ is a left Kan extension from a small full subcategory $\mathcal{D}$ of $\mathcal{M}$, hence a weighted colimit of representable functors, which, in turn, may be viewed as a coequalizer, by the dual of [17, 3.68]:

$$F = \int^{\mathcal{D}} R_D \otimes FD = \mathrm{coeq}\left(\coprod_{f\colon D' \to D} R_{D'} \otimes FD \rightrightarrows \coprod_{D} R_D \otimes FD \right).$$

Therefore, every $\mathcal{V}$-natural transformation $R_A \to F$ in $\left(\mathcal{V}^{\mathcal{M}^{\mathrm{op}}}\right)_0$ factors through $R_D \otimes FD$ for some $D \in \mathcal{D}$ by the weak Yoneda lemma for $\mathcal{V}$-categories. Unfortunately, $R_D \otimes FD$ is not necessarily $\mathcal{I}$-cofibrant. However, we can find an $\mathcal{I}$-cofibrant object U having a factorization $R_A \to U \to R_D \otimes FD$ for every cofibrant $A \in \mathcal{M}$.

Let $q\colon \tilde{D} \xrightarrow{\sim} D$ be a cofibrant replacement in $\mathcal{M}$, and $U := R_{\tilde{D}} \otimes \tilde{D}$, with the map $U \to R_D \otimes FD$ composed of $R_D \otimes q$ and $\hom(-, q) \otimes \tilde{D}$. It suffices to show that the induced map $\hom(R_A, U) \to \hom(R_A, R_D \otimes FD)$ is an epimorphism. This map factors, in turn, as a composition of two maps $\hom(A, \tilde{D}) \otimes \tilde{D} \to \hom(A, D) \otimes \tilde{D} \to \hom(A, D) \otimes D$, each of which is given by tensoring an effective epimorphism with an object of $\mathcal{V}$. A map is an effective epimorphism if and only if it is the coequalizer of some pair of parallel maps (see, e.g., the dual of [15, 10.9.4]). Hence, this is a composition of two effective epimorphisms, and so an epimorphism. In particular, if S is a unit of $\mathcal{V}$, then $\hom_{\mathcal{V}_0}(S, \hom(R_A, U)) \to \hom_{\mathcal{V}_0}(S, \hom(R_A, R_D \otimes FD))$ is a surjection of sets.

$\mathcal{V}^{\mathcal{M}^{\mathrm{op}}}$ becomes a $\mathcal{V}$-model category by [4, Prop. 3.18]. $\square$

Definition 2.4. Consider the following classes of maps in $\mathcal{V}^{\mathcal{M}^{op}}$:

$$\mathcal{F}_0 = \left\{ \emptyset = \operatorname*{hocolim}_{\emptyset} \emptyset \to R_{\operatorname{hocolim}_\emptyset \emptyset} = R_\emptyset \right\}$$

$$\mathcal{F}_1 = \{ R_X \otimes A \to R_{X \otimes A} |\, X \in \mathcal{M},\ A \in \mathcal{V} - \text{cofibrant objects} \} \ .$$

where the map $R_X \otimes A \to R_{X\otimes A}$ is the unit of the adjunction $Y\colon \mathcal{M} \leftrightarrows \mathcal{V}^{\mathcal{M}^{\mathrm{op}}} : - \otimes \mathrm{Id}_{\mathcal{M}}$.

$$\mathcal{F}_2 = \left\{ \operatorname{hocolim} \begin{array}{ccc} R_A & \longrightarrow & R_B \\ \downarrow & & \\ R_C & & \end{array} \to R_D \ \middle|\ \begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array} \quad \begin{array}{c} -\text{ homotopy pushout of} \\ \text{cofibrant objects in } \mathcal{M} \end{array} \right\}$$

$$\mathcal{F}_3 = \left\{ \begin{array}{c} \operatorname{hocolim}_{k<\kappa}(R_{A_0} \to \ldots \to R_{A_k} \to R_{A_{k+1}} \to \ldots) \\ \downarrow \\ R_{\operatorname{colim}_{k<\kappa} A_k} \end{array} \ \middle|\ \begin{array}{c} A_0 \hookrightarrow \ldots \hookrightarrow A_k \hookrightarrow A_{k+1} \hookrightarrow \ldots, \\ A_k \text{ cofibrant for all } k < \kappa \end{array} \right\}$$

Set $\mathcal{F} := \mathcal{F}_0 \cup \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$.

Definition 2.5. Given a class $\mathcal{F}$ of natural transformations of cofibrant-projectively cofibrant small functors $\mathcal{M}^{\mathrm{op}} \to \mathcal{V}$ (for $\mathcal{M}$ and $\mathcal{V}$ as in Theorem 2.3), we say that a functor $F : \mathcal{C} \to \mathcal{V}$ is *$\mathcal{F}$-local* if every $f : G \to H$ in $\mathcal{F}$ induces a weak equivalence $f^*\colon \hom(H, F) \to \hom(G, F)$.

Proposition 2.6. *Let $\mathcal{M}$ be a $\mathcal{V}$-model category, and assume that the underlying category of the category of small functors $\mathcal{V}^{\mathcal{M}^{\mathrm{op}}}$ may be equipped with the cofibrant projective model structure. Let $F \in \mathcal{V}^{\mathcal{M}^{\mathrm{op}}}$ be a cofibrant-projectively fibrant small functor taking weighted homotopy colimits of cofibrant objects to homotopy limits in $\mathcal{V}$. Then the functor F is $\mathcal{F}$-local (with respect to the class $\mathcal{F}$ of maps from Definition 2.4).*

Proof. This follows from Yoneda's lemma and the fact that the colimit of a sequence of cofibrations of cofibrant objects is a homotopy colimit. $\square$

Remark 2.7. We have only included in the class $\mathcal{F}$ those morphisms which are required for the proof of the inverse implication: $\mathcal{F}$-local functors are equivalent to the representable functors (see Theorem 3.1). In some situations the class $\mathcal{F}$ of maps may be reduced even further. For example, if $\mathcal{V} = \mathcal{S}$, then the subclass $\mathcal{F}_1$ of maps is redundant, since every weighted homotopy colimit in a simplicial category can be expressed in terms of the classical homotopy colimits (cf. [8, Lemma 3.1]).

If $\mathcal{V} = \mathcal{M} = \mathrm{Sp}$ for some closed symmetric monoidal combinatorial model of spectra, again $\mathcal{F}_1$ is not needed, by to Spanier-Whitehead duality (see [4, Lemma 7.2]).

However, in general weighted homotopy colimits cannot be expressed in terms of classical homotopy colimits (that is, homotopy colimits with contractible weight), as is shown by Lukáš-Vokřínek in [27].

3. Freyd representability theorem, up to homotopy

Theorem 3.1. *Let $\mathcal{V}$ be a closed symmetric monoidal combinatorial model category satisfying condition 2.2, and suppose that the domains of the generating cofibrations of $\mathcal{V}$ are cofibrant. If $\mathcal{M}$ is a $\mathcal{V}$-model category, then the underlying category of the category $\mathcal{V}^{\mathcal{M}^{\mathrm{op}}}$ of small $\mathcal{V}$-presheaves on $\mathcal{M}$ may be equipped with the cofibrant-projective model structure. A small functor F is cofibrant-projectively weakly equivalent to a representable functor if and only if it takes homotopy colimits of cofibrant objects to homotopy limits.*

Proof. Let λ be a regular cardinal such that $\mathcal{V}$ is a λ-combinatorial model category. Let $\mathcal{I}$ and $\mathcal{J}$ be its sets of generating cofibrations and trivial cofibrations, respectively. We assume that the domains of the maps in $\mathcal{I}$ are cofibrant.

Given a small functor $F \in \mathcal{V}^{\mathcal{M}^{\mathrm{op}}}$ taking homotopy colimits to homotopy limits, let $\widetilde{F} \tilde{\twoheadrightarrow} F$ be a cofibrant replacement for F in the cofibrant-projective model structure. Then there is a λ-sequence $F_0 \to F_1 \to \cdots \to F_k \to F_{k+1} \to \cdots \widetilde{F}$ such that $F_0(M) = \emptyset$ for all $M \in \mathcal{M}$, $\widetilde{F} = \operatorname{colim} F_k$, and F_{k+1} is obtained from F_k as a pushout

$$\begin{tikzcd} R_M \otimes A \arrow[r] \arrow[d, hook] & F_k \arrow[d] \\ R_M \otimes B \arrow[r] & F_{k+1}, \end{tikzcd}$$

where $M \in \mathcal{M}$ is cofibrant, and $(A \hookrightarrow B) \in \mathcal{I}$, A, $B \in \mathcal{V}$ are also cofibrant by assumption.

Our proof will proceed by induction. Recall the class $\mathcal{F}$ of maps from Definition 2.4, and note that the fibrant replacement of the given functor F is $\mathcal{F}$-local by Proposition 2.6.

Note also that $F_0 = \emptyset$ is $\mathcal{F}$-equivalent to $R_\emptyset = R_{\mathrm{hocolim}_\emptyset \emptyset}$, since the map $\emptyset \to R_\emptyset$ is in $\mathcal{F}_0 \subset \mathcal{F}$.

Suppose by induction that F_k is $\mathcal{F}$-equivalent to a representable functor R_{X_k}, where X_k is a fibrant and cofibrant object of $\mathcal{M}$. There is then a commutative diagram

$$\begin{tikzcd}
 & & R_{X'_k} \arrow[dr, two heads, "\sim"] & \\
R_{M\otimes A} \arrow[urr] \arrow[rrr] \arrow[dddd] & & & R_{X_k} \\
 & R_M \otimes A \arrow[ul, "\varepsilon"'] \arrow[r] \arrow[d, hook] & F_k \arrow[uu] \arrow[ur] \arrow[d] & \\
 & R_M \otimes B \arrow[r] \arrow[dl] & F_{k+1} & \\
R_{M\otimes B} & & &
\end{tikzcd}$$

where the upper horizontal arrow is induced by the universal property of the unit of the adjunction, and X'_k is a fibrant and cofibrant object of $\mathcal{M}$ obtained as a middle term of the factorization $M \otimes A \hookrightarrow X'_k \overset{\sim}{\twoheadrightarrow} X_k$.

Since the functor F_k is cofibrant, there exists a lift $F_k \to R_{X'_k}$ which is an $\mathcal{F}$-equivalence by the 2-out-of-3 property and does not violate the commutativity of the above diagram, since the upper slanted arrow $R_{M\otimes A} \to R_{X'_k}$ is also a natural map induced by the universal property of the unit of adjunction ε.

Let $P := M \otimes B \coprod_{M\otimes A} X'_k$. We then obtain a commutative diagram

$$\begin{tikzcd}
R_{M\otimes A} \arrow[rrr] \arrow[ddd] & & & R'_{X_k} \arrow[ddd] \\
 & R_M \otimes A \arrow[ul] \arrow[r] \arrow[d, hook] & F_k \arrow[ur] \arrow[d] & \\
 & R_M \otimes B \arrow[r] \arrow[dl] & F_{k+1} \arrow[dr, dashed] & \\
R_{M\otimes B} \arrow[rrr] & & & R_P
\end{tikzcd}$$

in which all the solid slanted arrows are $\mathcal{F}$-equivalences and the inner square is a homotopy pushout. Hence, the homotopy pushout of the outer square is $\mathcal{F}$-equivalent to F_{k+1}, and thus the dashed arrow is also an $\mathcal{F}$-equivalence. Finally, let X_{k+1} denote the middle element in the factorization $X'_k \hookrightarrow X_{k+1} \overset{\sim}{\twoheadrightarrow} \hat{P}$. Then $R_P \to R_{\hat{P}}$ is also an $\mathcal{F}$-equivalence, and thus by the 2-out-of-3 property, so is the lift $F_{k+1} \to R_{X_{k+1}}$ (which exists since F_{k+1} cofibrant).

If κ is a limit ordinal, then $F_\kappa = \operatorname{colim}_{k<\kappa} F_k$. Since this is the colimit of a sequence of cofibrations of cofibrant functors, $\operatorname{colim}_{k<\kappa} F_k$ is the homotopy colimit $\operatorname{hocolim}_{k<\kappa} F_k$. However, F_k is $\mathcal{F}$-equivalent to R_{X_k}. Moreover, by construction there is a sequence of cofibrations $X_0 \hookrightarrow \ldots \hookrightarrow X_k \hookrightarrow X_{k+1} \hookrightarrow \ldots$. Hence $\operatorname{hocolim} R_{X_k}$ is $\mathcal{F}$-equivalent to $R_{\operatorname{colim} X_k}$.

If κ is not large enough to ensure that $\operatorname{colim}_{k<\kappa} X_k$ is fibrant, we can consider the fibrant replacement $\operatorname{colim}_{k<\kappa} X_k \overset{\sim}{\hookrightarrow} \widehat{\operatorname{colim}_{k<\kappa}} X_k = X_\kappa$. Combining these facts together we conclude that F_κ is $\mathcal{F}$-equivalent to a representable functor R_{X_κ}, represented by a fibrant and cofibrant object.

For κ large enough we have $F = F_\kappa$, so F is $\mathcal{F}$-equivalent to a functor represented by a fibrant and cofibrant object. But F is an $\mathcal{F}$-local functor by Proposition 2.6, and so is R_{X_κ} for every κ. Hence, $F \simeq R_{X_\kappa}$ for some κ, since an $\mathcal{F}$-equivalence of $\mathcal{F}$-local functors is a weak equivalence. □

The formal category theoretic dual of Theorem 3.1 is the following:

Theorem 3.2. *Let $\mathcal{V}$ be a closed symmetric monoidal combinatorial model category satisfying condition 2.2, and let $\mathcal{M}$ be a $\mathcal{V}$-model category such that the category $\left(\mathcal{V}^{\mathcal{M}}\right)_0$ may be equipped with the fibrant-projective model structure. A small functor $F\colon \mathcal{M}^{\mathrm{op}} \to \mathcal{V}$ is then fibrant-projectively weakly equivalent to a representable functor if and only if it takes homotopy limits of fibrant objects to homotopy limits.*

4. Brown representability theorem, up to homotopy

In this section we prove a homotopy version of the Brown Representability Theorem for contravariant functors from a locally presentable $\mathcal{V}$-model category $\mathcal{M}$ to $\mathcal{V}$ taking homotopy colimits to homotopy limits.

Note that our proof does not use explicitly the presence of compact objects, as most known proofs in this field do. Rather we show directly that a contravariant homotopy functor from $\mathcal{M}$ to $\mathcal{V}$ is cofibrant projectively weakly equivalent to a small functor, and then use the Freyd representability theorem, up to homotopy.

Lemma 4.1. *Let $\mathcal{M}$ be a combinatorial $\mathcal{V}$-category. Then any functor $H\colon \mathcal{M}^{\mathrm{op}} \to \mathcal{V}$ taking homotopy colimits of cofibrant objects to homotopy limits is cofibrant-projectively weakly equivalent to a small homotopy functor $F \in \mathcal{V}^{\mathcal{M}^{\mathrm{op}}}$.*

Proof. Suppose $\mathcal{M}$ is a λ-combinatorial model category, for some cardinal λ such that the weak equivalences are a λ-accessible subcategory of the category of maps of $\mathcal{M}$.

Consider a functor $H\colon \mathcal{M}^{\mathrm{op}} \to \mathcal{V}$ taking homotopy colimits of cofibrant objects to homotopy limits, with the natural map

$$H \to F = \mathrm{Ran}_{i\colon \mathcal{M}_\lambda \hookrightarrow \mathcal{M}}\, i^*H \ ,$$

where Ran is the right Kan extension along the inclusion of categories $i\colon \mathcal{M}_\lambda \hookrightarrow \mathcal{M}$. Here $\mathcal{M}_\lambda$ denotes the full subcategory of λ-presentable objects in $\mathcal{M}$.

This map is a weak equivalence in the cofibrant-projective model category, because both functors take homotopy colimits to homotopy limits and every cofibrant object of $\mathcal{M}$ is a (λ-filtered) homotopy colimit of λ-presentable cofibrant objects (see [19, Cor. 5.1]), on which the two functors coincide.

But we can interpret the right Kan extension as a weighted inverse limit of representable functors

$$F = \mathrm{Ran}_{i\colon \mathcal{M}_\lambda \hookrightarrow \mathcal{M}}\, i^*H = \int_{M \in \mathcal{M}_\lambda} \hom(H(M), M) = \{i^*H, i^*Y\},$$

where $Y\colon \mathcal{M} \to \mathcal{V}^{\mathcal{M}^{\mathrm{op}}}$ is the Yoneda embedding and $\{i^*H, i^*Y\}$ is the weighted inverse limit of i^*Y indexed by i^*H (see [17, 3.1]).

Then by the theorem of Day and Lack in [10], F is small as an inverse limit of small functors.

□

Remark 4.2. When we speak about sufficiently large filtered colimits in a combinatorial model category, they turn out to be homotopy colimits, no matter if the objects participating in them are cofibrant or not. Therefore the above Lemma admits the following concise formulation: any functor taking homotopy colimits to homotopy limits is levelwise weakly equivalent to a small functor.

Theorem 4.3. *Let $\mathcal{V}$ be a closed symmetric monoidal model category satisfying condition 2.2, and $\mathcal{M}$ a combinatorial $\mathcal{V}$-model category. Any functor $H\colon \mathcal{M}^{\mathrm{op}} \to \mathcal{V}$ taking homotopy colimits of cofibrant objects to homotopy limits is then cofibrant-projectively weakly equivalent to a representable functor.*

Proof. Follows from Theorem 3.1 by Lemma 4.1. □

5. Counter-example

We have proved so far two representability theorems, up to homotopy. The first is of Freyd type, i.e., some set theoretical conditions are required to be satisfied by the functor in question. The second one is of Brown type, i.e., some set theoretical assumptions apply to the domain category. But what happens if we make no set theoretical assumptions on either the domain category or the functor? Are exactness conditions enough to ensure representability up to homotopy?

Mac Lane's classical (folklore) example of a functor $B\colon \mathrm{Grp} \to \mathrm{Set}$, which assigns to each group G the set of all homomorphisms from the free product of a large collection of non-isomorphic simple groups to G, is an example of a (strictly) non representable functor. Notice that neither does B satisfy the solution set condition, nor is the category $\mathrm{Grp}^{\mathrm{op}}$ locally presentable. Perhaps representability up to homotopy is less demanding and would persist without any conditions?

Our example is similar in nature to Mac Lane's example, but it has also another predecessor: in [11], Dror-Farjoun gave an example of a failure of Brown representability for generalized Bredon cohomology.

Consider the closed symmetric combinatorial model category of spaces $\mathcal{S}$, with $\mathcal{M} = \mathcal{S}^{\mathcal{S}}$ the category of small functors. Let $B\colon \mathcal{S} \to \mathcal{S}$ be a functor which is not small, such as $B = \hom(\hom(-, S^0), S^0)$. Note that $B \notin \mathcal{M}$, since B is not accessible and all small functors are. However, $H = \hom(-, B)\colon \mathcal{M} \to \mathcal{S}$ is well defined, since for any small functor $F \in \mathcal{M}$, there exists a small subcategory $i\colon \mathcal{A} \hookrightarrow \mathcal{S}$ such that $F = \mathrm{Lan}_i\, i^*F$ and hence $H(F) = \hom(F, B) = \hom_{\mathcal{S}^{\mathcal{A}}}(i^*F, i^*B) \in \mathcal{S}$.

We have defined a functor $H\colon \mathcal{M}^{\mathrm{op}} \to \mathcal{S}$ taking homotopy colimits of cofibrant objects to homotopy limits, but it is not representable, even up to homotopy: otherwise, there would exist a small fibrant functor $A \in \mathcal{M}$ such that $H(-) \simeq \hom(-, A)$. By J. H. C. Whitehead's argument we know $A \simeq B$, but then B would preserve the λ-filtered homotopy colimits for some λ. This is a contradiction, since B does not preserve filtered colimits even of discrete spaces.

6. Mapping space recognition principle

In this section we assume that the closed symmetric monoidal combinatorial model category $\mathcal{V}$ is the category $\mathcal{T}\mathrm{op}_*$ of Δ-generated topological spaces. This is a locally presentable version of the category of topological spaces, first proposed by Jeff Smith and described in detail by Fajstrup and Rosicky in [12].

Let $X \in \mathcal{T}\mathrm{op}_*$ be a path-connected space, and let $A \in \mathcal{T}\mathrm{op}_*$ be a CW-complex. We will describe a sufficient condition for there to exist a space $Y \in \mathcal{T}\mathrm{op}_*$ such that $X \simeq \hom(A, Y)$. For example, if $A = S^1$, the required condition is that X can be equipped with an algebra structure over the little intervals operad. In practice that means that the space X admits k-ary operations, i.e., maps $X \times \ldots \times X \to X$ satisfying a long list of higher associativity conditions.

For a space A more general then S^1 it is insufficient to consider the structure given by the maps $X \times \ldots \times X \to X$, by [3]. We will consider, instead, the structure given by the mutual interrelations of all possible homotopy inverse limits of X – the structure we would have if X indeed was equivalent to $\hom(A, Y)$ for some Y. Indeed, consider the subcategory of A-cellular spaces $C(A) \subset \mathcal{T}\mathrm{op}_*$. This is the minimal subcategory containing A and closed under the homotopy colimits. Any homotopy colimit of a diagram involving A is then taken by the functor $\hom(-, Y)$ into the homotopy limit of an opposite diagram involving X. In other words, every mapping space X is equipped with a functor $F_X : C(A) \to \mathcal{T}\mathrm{op}_*$. Moreover, this functor has a very nice property: it takes homotopy colimits into homotopy limits.

Our goal is to show the converse statement: if for a given $X \in \mathcal{T}\mathrm{op}_*$ there exists a functor $F : C(A)^{\mathrm{op}} \to \mathcal{T}\mathrm{op}_*$ taking homotopy colimits to homotopy limits and satisfying $F(A) \simeq X$, this F is weakly equivalent to a representable functor, i.e., there is a space Y such that $F(-) \simeq \hom(-, Y)$.

Theorem 6.1. *For any cofibrant $A \in \mathcal{T}\mathrm{op}_*$ and any $X \in \mathcal{T}\mathrm{op}_*$, there is an object $Y \in \mathcal{T}\mathrm{op}_*$ satisfying $X \simeq \hom(A, Y)$ if and only if there exists a functor $F : C(A)^{\mathrm{op}} \to \mathcal{T}\mathrm{op}_*$ taking homotopy colimits to homotopy limits and satisfying $F(A) \simeq X$.*

Proof. Necessity of the condition is clear. We will prove the sufficiency now.

Consider the right Bousfield localization of $\mathcal{T}\mathrm{op}_*$ with respect to A. By [15, 5.1.1(3)] we obtain a cofibrantly generated model structure, which is also combinatorial, since we have chosen to work with a locally presentable model of topological spaces. We denote the new model category by $\mathcal{T}\mathrm{op}_*^A$. The subcategory of cofibrant objects of $\mathcal{T}\mathrm{op}_*^A$ is then $C(A)$ as above. We denote the inclusion functor by $i : C(A) \to \mathcal{T}\mathrm{op}_*^A$.

Given a functor $F : C(A) \to \mathcal{T}\mathrm{op}_*$ taking homotopy colimits to homotopy limits and satisfying $F(A) = X$, consider the left Kan extension $H = \mathrm{Lan}_i F$ of F along the inclusion i. This functor $H : \mathcal{T}\mathrm{op}_*^A \to \mathcal{T}\mathrm{op}_*$ then satisfies the condition of Theorem 4.3, hence there exists $Y \in \mathcal{T}\mathrm{op}_*^A$ such that $H(-) \simeq \hom(-, Y)$, in particular, $H(A) = F(A) \simeq X \simeq \hom(A, Y)$. □

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